THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

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LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN, AND BOMBAY.

THE ANNUAL GENERAL MEETING.

THE Annual General Meeting of the Mathematical Association was held at King's College, London, on Saturday, January 19th, at two o'clock. The retiring President, Sir Robert S. Ball, LL.D., F.R.S., occupied the chair. Twenty members were present.

Mr. J. Fletcher Moulton, K.C., M.P., F.R.S., was elected President. Sir Robert S. Ball and Professor W. H. H. Hudson were elected Vice-Presidents; and Mr. C. E. Williams, M.A., was elected a Member of the Council.

The election of twenty-one new members of the Association was confirmed.

It was decided that members might compound for three years' subscriptions by the payment of £1 1s. in advance; the composition fee for life membership still remaining at £5 5s.

After the Treasurer had given his report, Sir Robert Ball made a communication entitled "Some Contributions to Geometry from recent Dynamical Work," in which a number of theorems of great interest and generality were explained. Professor M. J. M. Hill, F.R.S., then read a paper on "The Teaching of Proportion in Geometry." Messrs. C. E. M'Vicker, E. Budden, F. S. Macaulay, W. H. H. Hudson, and P. J. Harding took part in the discussion which followed; and Professor Hill replied to the questions which had been raised. The meeting closed at five o'clock.

ON A GEOMETRICO-STATICAL THEOREM.

1. Let A_1, A_2, A_3, A_4 be four points ranged in this order along the circumference of a horizontal circular lamina: and at A_1, A_2, A_3, A_4, let vertical forces

$$+1/A_1A_2 \,.\, A_1A_3 \,.\, A_1A_4, \qquad -1/A_2A_1 \,.\, A_2A_3 \,.\, A_2A_4,$$
$$+1/A_3A_1 \,.\, A_3A_2 \,.\, A_3A_4, \qquad -1/A_4A_1 \,.\, A_4A_2 \,.\, A_4A_3,$$

be respectively applied (the forces at A_1, A_3 acting downwards

and those at A_2, A_4 acting upwards),—then this system of parallel forces will be in equilibrium.

For taking moments about A_1A_2

$$\{1/A_3A_1 . A_3A_2 . A_3A_4\} . A_3A_1 . A_3A_2 \\ -\{1/A_4A_1 . A_4A_2 . A_4A_3\} . A_4A_1 . A_4A_2 = 0,$$

since $A_3A_1 . A_3A_2$ is proportional to the perpendicular from A_3 on A_1A_2.

In like manner, the moment round A_2A_3 or A_3A_4 vanishes; and, the moment vanishing round three non-intersecting lines, the forces are in equilibrium.

2. Now instead of four points we will suppose $A_1, A_2, \dots A_n$ are any n points ranged in this order round the circumference (n is EVEN and >2), at which forces $P_1, P_2, \dots P_n$ respectively act, where $$P_1 = 1/A_1A_2 . A_1A_3 \dots A_1A_n$$ and similarly for the rest, those forces at the odd points acting vertically downwards and those at the even points upwards. Assuming it is true that these forces are in equilibrium

$$(P_1 + P_3 + \dots + P_{n-1} = P_2 + P_4 + \dots + P_n, \text{ etc.})$$

we can then prove that the same property holds true for $n+2$ points.

3. For taking two additional points A_{n+1}, A_{n+2} between A_n and A_1, suppose a new system of forces $Q_1, Q_2, \dots Q_{n+2}$ applied respectively such that the force

$$Q_1 = 1/A_1A_2 . A_1A_3 \dots A_1A_n . A_1A_{n+1} . A_1A_{n+2},$$

and similarly for the rest. Then taking moments round the chord $A_{n+1}A_{n+2}$ the moment of Q_1 is proportional to P_1

(since $A_1A_{n+1} . A\ A_{n+2} \propto$ perpendicular from A_1 on the chord)

and the total moment $\propto P_1 + P_3 + \dots + P_{n-1} - P_2 - P_4 \dots - P_n$, which vanishes by assumption.

In like manner the moment vanishes round any other two sides of the polygon and the forces are in equilibrium.

The theorem, therefore, being true for the quadrilateral, is also true for any even number of points.

4. It now remains to introduce the point O in space whose coordinates are x, y, z referred to any rectangular axes through the centre of the circle (radius a);

$$\begin{aligned}\Sigma P_1 . OA_1^2 &= \Sigma P_1\{(x - a\cos\theta_1)^2 + (y - a\sin\theta_1)^2 + z^2\} \\ &= (x^2 + y^2 + z^2 + a^2)\Sigma P_1 - 2ax\Sigma P_1\cos\theta_1 - 2ay\Sigma P_1\sin\theta_1 \\ &= 0\end{aligned}$$

since $\Sigma P_1 = \Sigma P_1 \cos\theta_1 = \Sigma P_1 \sin\theta_1 = 0$ as the forces are in equilibrium. Thus $\Sigma \pm OA_1^2/A_1A_2 \dots . A_1A_n = 0$, the terms being alternately plus and minus.

5. It will thus be seen that the theorem proposed as Question 390 in the *Mathematical Gazette*, No. 24, December, 1900 (and proved by co-reciprocal screws at the meeting of the Association on January 19th), is true not only for the hexagon but for any cyclic polygon of an even number of sides.

6. It has been kindly brought to my notice by Mr. R. F. Davis that Salmon in his *Conics* (end of Chap. VI. on the circle) gives

$$OA^2 . \triangle BCD + OC^2 . \triangle ABD = OB^2 . \triangle ACD + OD^2 . \triangle ABC,$$

which may be written $\Sigma \pm OA^2/AB . AC . AD = 0$.

7. A concise statement of the general theorem is

$$\Sigma(-1)^\kappa . \lambda_\kappa . OA_\kappa{}^2 = 0,$$

where λ_κ is the reciprocal of the continued product of the chords through A_κ.

R. S. BALL.

REVIEW.

L'Elimination. Par H. LAURENT. Pp. 75. *Scientia*, Mars, 1900. Carré et Naud. Paris.

In his preface the author states that no monograph has been published on the theory of elimination since one by Fàa de Bruno in 1859. The subject is undoubtedly treated without sufficient care in the ordinary English text-books on Algebra; and, according to Laurent, there is no continental work which contains a complete account of all the recent developments.

The book starts with a number of familiar properties of symmetric functions; these are at once applied to obtain various methods for finding the resultant (defined in § 4)[1] of two equations ($\phi=0$, $\psi=0$) in a single variable. Specially interesting is Cauchy's method (§ 8); but, unfortunately, this contains some misleading printer's errors, notably in the proof that the resultant is of the form $(\lambda\psi - \mu\phi)$. Speaking of this, it is worth while to point out that the printer has frequently confused suffixes and indices throughout the book; and, in some places, when a product should occur at the end of a line, the two elements have been separated, one being placed at the end of the line while the other is put at the beginning of the next line.

A little more use might be made of geometrical illustrations; for example, Bezout's theorem (§§ 12, 16, 17) may be stated thus: Two co-planar curves of degrees m, n meet in mn points; and, of course, a similar theorem holds for three surfaces in space. Another illustration may be found in § 13; a double solution corresponds to a point of contact of the curves $\phi=0$, $\psi=0$; and the exceptional case is represented by a curve which passes through a double point of the other curve.

In Chapter II. the author deals with the general problem of eliminating n variables from $(n+1)$ equations; and starts with an account of systems of moduli (Kronecker's *Modulsystem*; see, for instance, Crelle's *Journal*, Bd. 99, p. 329). He then obtains a theorem[2] due to Jacobi (§ 19), and, after some intermediate steps, gives the method for determining the resultant in § 23, with some properties of the resultant in § 24. Ləbatie's method for two equations (by means of the H.C.F. process) is indicated in § 25, where use might be made of the theory of continued fractions.

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[1] Here $(-1)^n$ should be $(-1)^{mn}$.

[2] This is an extension of the familiar theorem

$$\Sigma \frac{g(a)}{f'(a)} = 0,$$

where $f(x)$ is a polynomial of degree m, a is a root of $f(x)=0$ (all the roots being different) and $g(x)$ is a polynomial of degree less than $(m-1)$.

Two examples of some importance are given in §§ 29, 30, and we shall make a few criticisms on them. The first (§ 29) is an extension of a well-known problem in Solid Geometry: It is known that the directions of the principal axes of the quadric $(abcfgh \between xyz)^2 = 1$ are given by

$$\frac{1}{l}(al+hm+gn)=\frac{1}{m}(hl+bm+fn)=\frac{1}{n}(gl+fm+cn),$$

and the corresponding principal planes by $lx+my+nz=0$; it is required to eliminate l, m, n so as to have an explicit equation for the principal planes. The resul of this elimination is generally given in the form

$$\begin{vmatrix} x, & ax+hy+gz, & Ax+Hy+Gz \\ y, & hx+by+fz, & Hx+By+Fz \\ z, & gx+fy+cz, & Gx+Fy+Cz \end{vmatrix} = 0.$$

Laurent obtains a result (for any number of variables) from which the above is deducible. There is, however, a slip in Laurent's work, which, fortunately, does not invalidate the method; but, as this slip is one which may be quite commonly made, it seems advisable to call attention to it. We have the $2n$ equations $f_r = \Sigma a_{rs}x_s$, $g_r = \Sigma b_{rs}x_s$ $(r, s = 1, 2, \dots, n)$ from which by solution $f_r = \Sigma c_{rs}g_s$. Then Laurent seems[1] to conclude that, if $a_{sr} = a_{rs}$ and $b_{sr} = b_{rs}$, it follows that $c_{sr} = c_{rs}$; this is not, however, true except under special circumstances. This fact will be recognized at once by readers acquainted with Cayley's theory of matrices[2] (or the theory of bilinear forms); but to give an elementary illustration take the case of two variables with $b_{11} = 0 = b_{22}$, $b_{12} = 1 = b_{21}$, then we have $g_1 = x_2$ and $g_2 = x_1$, thus we find $f_1 = a_{12}g_1 + a_{11}g_2$, $f_2 = a_{22}g_1 + a_{12}g_2$; and here the conditions are satisfied by the a's and b's, but not by the c's, unless $a_{11} = a_{22}$.

In § 30 Laurent discusses the important determinantal equation

$$\Delta = |f_{rs} + \lambda g_{rs}| = 0, \qquad (f_{rs} = f_{sr},\ g_{rs} = g_{sr})$$

which occurs in the theory of the two quadratic forms $f = \Sigma f_{rs}x_r x_s$, $g = \Sigma g_{rs}x_r x_s$. At the top of p. 61 we find a theorem included in Kummer's[3] that the discriminant of Δ can be expressed as a sum of squares, provided that one of the forms f, g is *definite* (*i.e.*, is always of one sign for all *real* values of the x's). At the foot of the same page it is stated that the general case may be reduced to that in which[4] $g = x_1^2 + x_2^2 + \dots + x_n^2$, but this is misleading, for it might be supposed that the results of p. 62 hold without any restriction on f, g; whereas they are only certainly true if one of the forms f, g is definite. The theorem alluded to is that if $(\lambda - a)^\alpha$ is a factor of Δ, then $(\lambda - a)^{\alpha - 1}$ is a factor of *all* the first minors of Δ; or, in Weierstrass's terminology, all the invariant-factors of Δ are linear. An illustration will show how the theorem may fail. Take the quadratic forms

$$f = -ag + 2yz, \quad g = 2xy + z^2,$$

then

$$\Delta = \begin{vmatrix} 0, & \lambda - a, & 0 \\ \lambda - a, & 0, & 1 \\ 0, & 1, & \lambda - a \end{vmatrix} = -(\lambda - a)^3,$$

but one first minor of Δ is -1; which shows that the theorem cannot hold for *all* pairs of quadratic forms. Laurent's proof of the theorem assumes (tacitly) that the roots of $\Delta = 0$ are all real; for, otherwise, the argument would fail.[5] This assumption is certainly correct provided that one of the forms f, g is definite, but it should, I think, have been explicitly stated. It may be worth while to point out, further, that this theorem is closely connected with Weierstrass's important

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[1] The exact statement is $c_{ij} = c_j$; presumably a printer's error.

[2] *Coll. Math. Papers*, vol. ii., p. 475; Laurent himself published a paper in Liouville's *Journal* (1898) where similar results are obtained.

[3] Crelle's *Journal*, Bd. 26, 1843, p. 268; *Salmon's Modern Higher Algebra*, Lesson vi., p. 55, 4th Ed.

[4] This reduction is always possible if the use of imaginary substitutions is admissible; but there is then no guarantee that $g = 0 = x_1^2 + x_2^2 + \dots + x_n^2$ really necessitates $x_1 = 0 = x_2 = \dots = x_n$ Thus, in the case quoted below, $g = 0$ can be satisfied by $x = 0$, $z = 0$ for *any* value of y.

[5] The proof shows that a sum of squares vanishes; and it is deduced that each vanishes separately. This is, of course, only true if all the quantities involved are real.

result[1] that *two quadratic forms f, g can always be reduced to sums of the same squares, whether $\Delta=0$ has repeated roots or not, provided that one of the forms f, g is definite.*

The book closes with a number of miscellaneous theorems and methods; the appendix contains a proof of the proposition that the solutions of a system of equations are continuous functions of the parameters which appear in the equations.

I have read this monograph with much interest and pleasure; but it would have added to my interest if more references had been given to the original authorities. A bibliography would also enable readers to follow up points indicated by original workers, which must necessarily be omitted in a connected account of the whole theory. T. J. I'A BROMWICH.

MATHEMATICAL NOTE.

94. [K. 10. e.] *The equation to the circumcircle of the triangle contained by three given straight lines.*

(1) Any circum-conic to the lines $u_1 \equiv \Sigma a_1 x = 0$ is of the form

$$\Sigma\lambda u_2 u_3 = 0. \quad\text{...(1)}$$

If this is a circle,

$$\Sigma\lambda(a_2b_3+a_3b_2)=0\ ; \quad \Sigma\lambda(a_2a_3-b_2b_3)=0. \quad\text{...........(2) and (3)}$$

Eliminating λ, μ, ν from (1), (2), and (3), we have for the equation to the circumcircle

$$\begin{vmatrix} u_2u_3, & u_3u_1, & u_1u_2 \\ a_2b_3+a_3b_2, & \dots & \dots \\ a_2a_3-b_2b_3, & \dots & \dots \end{vmatrix} = 0.$$

(2) To see what the actual form is, take $u_1 \equiv x\cos a_1 + y\sin a_1 - p_1 = 0$, etc. The equation to the circle is

$$\Sigma\lambda(x\cos a_2 + y\sin a_2 - p_2)(x\cos a_3 + y\sin a_3 - p_3) = 0.$$

where $\qquad \Sigma\lambda\cos(a_2+a_3) = \Sigma\lambda\sin(a_2+a_3) = 0,$

and as above, $\qquad \lambda/\sin(a_2-a_3) = \text{etc.}$

As only the ratios of λ, μ, ν are involved, we may take $\sin(a_2-a_3)$, etc., as their actual values.

Then, the coefficient

of x^2 is $\Sigma\sin(a_2-a_3)\cos a_2\cos a_3 = -\Pi\sin(a_2-a_3)$,

of x is $-\Sigma p_1(\mu\cos a_3+\nu\cos a_2) = \Sigma p_1\cos(a_2+a_3-a_1)\sin(a_2-a_3)$,

of y $\qquad = \Sigma p_1\sin(a_2+a_3-a_1)\sin(a_2-a_3)$.

The absolute term $\qquad = \Sigma p_2p_3\sin(a_2-a_3)$.

But $$\frac{\cos a_1}{a_1} = \frac{\sin a_1}{b_1} = \frac{p_1}{-c_1} = \frac{1}{\sqrt{a_1^2+b_1^2}} = \frac{1}{\sqrt{k_1}} \text{ (say)},$$

and the above becomes:

coeff. of $x^2 = \Pi(a_2b_3-a_3b_2)/(k_1k_2k_3)$,

.... $x = \Sigma c_1(a_2b_3-a_3b_2)(a_1a_2a_3 - a_1b_2b_3 + a_2b_3b_1 + a_3b_1b_2)/(k_1k_2k_3)$,

.... $y = \Sigma c_1(a_2b_3-a_3b_2)(b_1b_2b_3 - b_1a_2a_3 + b_2a_3a_1 + b_3a_1a_2)/(k_1k_2k_3)$,

absolute term $= -\Sigma c_2c_3(a_2b_3-a_3b_2)/(k_2k_3)$,

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[1] The full importance of this theorem will probably be seen after noticing that two forms such as $2yz$, $2xy+z^2$ *cannot* be reduced to sums of the same squares.

and, finally, the equation to the circle is

$$(x^2+y^2)\Pi(a_2b_3-a_3b_2)+x\Sigma c_1(a_2b_3-a_3b_2)(a_1a_2a_3-a_1b_2b_3+a_2b_3b_1+a_3b_1b_2)$$
$$+y\Sigma c_1(a_2b_3-a_3b_2)(b_1b_2b_3-b_1a_2a_3+b_2a_3a_1+b_3a_1a_2)$$
$$-\Sigma c_2c_3(a_1^2+b_1^2)(a_2b_3-a_3b_2)=0.$$

(3) Applying this to the case of the triangle, sides $x-m_1y+am_1^2=0$, etc. circumscribing the parabola $y^2=4ax$, we have $a_1=1$, $b_1=-m$, $c_1=am_1^2$.

coeff. of $x^2=\Pi(m_2-m_3)$; coeff. of x is $-a\Pi(m_2-m_3)(1+\Sigma m_2m_3)$;

coeff. of $y=-a\Pi(m_2-m_3)(\Sigma m_1-m_1m_2m_3)$;

absolute term is $a^2\Pi(m_2-m_3)\Sigma m_2m_3$;

and the equation to the circumcircle is

$$x^2+y^2-ax(1+\Sigma m_2m_3)-ay(\Sigma m_1-m_1m_2m_3)+a^2\Sigma m_2m_3=0.$$

(Cf. *Mess. Math.*, No. 352, Aug., 1900.)

(4) In the case of numerical examples it is better to proceed *ab initio*.

Find the centre of the circumcircle of the triangle formed by the lines

$x+2y+3=0$; $2x+3y+1=0$; $3x+y+2=0$. (St. Cath.'s, '96.)

The circle is $\Sigma\lambda(2x+3y+1)(3x+y+2)=0$,

where $6\lambda+3\mu+2\nu=3\lambda+2\mu+6\nu$ and $11\lambda+7\mu+7\nu=0$,

i.e. $$\frac{\lambda}{7}=\frac{\mu}{-13}=\frac{\nu}{2}=\frac{6\lambda+3\mu+2\nu}{7},$$

and the equation is $7(x^2+y^2)-80x-20y-58=0$.

Again, show that the circumcircle of the triangle formed by the lines

$$x\cos\theta+y\sin\theta=a\sec\theta+b\sin\theta\quad(\theta=\alpha,\ \beta,\ \gamma)$$

passes through the point $(0, b)$. (St. Cath.'s, '99.)

Changing the origin to $(0, b)$, the sides become

$$x\cos\theta+y\sin\theta=a\sec\theta\quad(\theta=\alpha,\ \beta,\ \gamma);$$

and the required condition is

$$\Sigma a^2\sec\beta\sec\gamma\sin\overline{\beta-\gamma}=0\quad\text{or}\quad\Sigma\cos\alpha\sin(\beta-\gamma)=0,$$

which is clearly true, E. M. RADFORD.

CORRESPONDENCE.

Professor Hill replies to Mr. Budden as follows:

I hope you will allow me a few words of reply to Mr. Budden's further criticisms on my edition of the Fifth and Sixth Books of Euclid.

Mr. Budden *assumes* that the ratio A/B admits of an arithmetical definition as a number when A and B are incommensurable. This is an attempt to evade the whole difficulty.

Mr. Budden's point of view is that which was generally accepted as correct prior to the publication of the purely arithmetic theories of the irrational number by Weierstrass, Cantor, and Dedekind in 1872.

In order to conform to the requirements of analysis at the present time, Mr. Budden must show that when A and B are incommensurable, the symbol or operator μ, appearing in the relation $A=\mu B$, admits of arith-

metical definition as a number, and so satisfies the associative, commutative, and distributive laws. This can of course be done, but the doing of it is far beyond the comprehension of beginners. Some idea of the difficulties involved may be formed by consulting the sketch of the subject given in the Second Edition of the Second Part of Chrystal's *Algebra*, pp. 97-106, together with the references he gives, especially to the works of Dedekind and Stolz. A good account by Pringsheim will also be found on pp. 49-57 of the first volume of the *Encyklopädie der Mathematischen Wissenschaften.* It is now so generally admitted that it is not permissible to assume that μ admits of arithmetical definition, and satisfies the fundamental laws of algebra, that it is not necessary to examine the rest of Mr. Budden's argument in detail.

It may not, however, be superfluous to notice two points.

The first is, that I am not entitled to the high honour of being the author of the three sets of conditions referred to on page 11 of the January number of the *Mathematical Gazette*, Art. 2, Note ii, as mine. These are the conditions stated in the Fifth Definition of the Fifth Book of Euclid. So far as I know, they were first reduced to two by Stolz in his *Vorlesungen über Allgemeine Arithmetik*, Part I., page 87, published in 1885.

The second is the reference in Mr. Budden's 5th Article to "Prof. Hill's system." On this point I desire to say that I have not invented a system. My work has been that of a commentator on Euclid, and I believe that I have accomplished two things. The first is the giving of adequate explanations of the definitions of the Fifth Book, which previously had to be learned by heart without being understood. The second is the removal of the indirectness from Euclid's line of argument *without departing from his principles*, by showing that all the properties of equal ratios can be deduced from the test for equal ratios (Euc. V., Def. 5) without the employment of the test for distinguishing between unequal ratios (Euc. V. Def. 7). The result is that the argument has been simplified to an extent which has brought it within the grasp of persons of average ability.

M. J. M. HILL.

[The reader must not assume that the Editor and his committee of co-operation agree with Mr. Budden's criticisms. The discussion has been inserted for the following reason. One aim of the *Gazette* is to keep teachers of Mathematics *au courant* with the advance of mathematical thought and methods. Mr. Budden and Professor Hill represent respectively two schools, the old and the new. With the object of showing teachers the difference between these schools, the Editor asked Professor Hill to reply to Mr. Budden's criticism. The reply involved an answer and a rejoinder. The controversy has been useful if for no other reason than that it has sent some of our readers to Stolz and Dedekind. Of course, the case *is summed up* in Professor Hill's "it is now so generally admitted that it is not permissible to assume. . . . "—W. J. G.]

NOTICES.

Eléments de Méthodologie Mathématique, by M. DAUZAT. Pp. 1100. 1900. (Nony, Paris.)

IF the agreeable anticipations to which the title of this volume gave rise have not been completely realised, it is no doubt because (pardon the profanity of the metaphor) there is not a one-to-one correspondence in text-books on both sides of the Channel. It is of course within the bounds of possibility that a book on Methodology in Mathematics may yet make a belated appearance in this country. But when it does, we will venture to say that, earnest and conscientious as is the voluminous compilation of M. Dauzat, an English imitator will not run to 1100 pages in covering the same ground.

For example, our author devotes 316 pages to the theory of Elementary Algebra (Progressions, Logarithms, and Interest being his wildest flight), whereas Chrystal's "Introduction to Algebra" deals far more rigorously with much more matter, and (omitting the exercises) in much the same space.

The book consists of 20 pages devoted to general considerations on the teaching of Elementary Mathematics, 326 dealing with the theory and practice of Arithmetic, about the same number dealing with Algebra, and some 400 treating of Plane and Solid Geometry, Geometrical Conics, Inversion, Reciprocation, Projection, etc., etc.

In his official capacity as *Inspecteur d'Académie* the author has been forced to the conclusion that, in general, neither teachers nor pupils "read with sufficient care the introductions to text-books, the preliminary considerations of the various chapters, the notes at the ends of solutions, the footnotes," and all that varied and invaluable apparatus of suggestion which may be said to be indispensable or extremely valuable, inasmuch as it casts a flood of light on felicitous correlation of ideas, on ingenious and fertile methods of comparison, on synthetic summaries, on generalisations, and in fact on every process that widens the intellectual horizon. And the object he has set before himself is to awaken and sustain the interest of master and pupil alike in methods of research, methods of proof, in the philosophy of elementary Mathematics, and in the logical interconnection of its various branches. "In aim and form my book is novel;[1] I have done my best to bring it 'up to date,' and I shall be very happy if it is found of use." This modest claim almost disarms criticism at the outset. The English reader will nevertheless find but little after the first twenty pages to arouse and sustain the attention until he reaches the section dealing with Geometry. Here, among many other excellent features, will be found discussions on various problems in elementary Geometry which are entirely admirable in their insight into difficulties and possibilities. To this we must add that the general remarks dealing with the teaching of elementary Mathematics and the aim it should have in view are excellent, and if generally adopted should spell "death" to rule-of-thumb methods. The object of such teaching is to "bring into play—attention, reflection, judgment, and the reasoning power. By developing and strengthening these faculties, it accustoms the student to clearness of thought and precision of statement; it contributes essentially to the arrangement in logical order of general ideas, to the detection of error, and to the search for truth; to the mind it gives accuracy and definite insight, and thus contributes in the most effective manner to general culture. Hence the teacher of Mathematics has the double task of furnishing and cultivating the mind. If he knows his subject and its methods, if he can inspire his pupils with a confidence in *him*, if he appeals only to their reasoning power, if he has faith and the sacred fire, he must succeed. To arouse and sustain a love for Mathematics, he must throw into relief its fascination, the wonderful and vigorous inter-dependence of mathematical truths, the power and simplicity of methods of research, and the opportunity open to all students of discovery—quite apart from the personal pleasure excited by such discovery. Order and regularity must be a part and parcel of the teacher's work; method and lucidity must prevail in every lesson, and every application of mathematical doctrine must be interesting, varied, and judiciously graduated."

The man who can write thus has the "sacred fire"; and after all, how many of those who profess and call themselves teachers can say as much?

A Brief History of Mathematics. Authorised translation of Dr. Karl Fink's *Geschichte der Elementar-Mathematik.* By W. W. Beman and D. E. Smith. Pp. xii. + 333. 1900. (Kegan Paul.)

To Dr. Karl Fink's able and conscientious work we can extend a hearty welcome. The historical literature of the subject in our language is scanty enough, and what we have is presented in a guise apparently inseparable from a somewhat chaotic mass of petty biographical detail. But Dr. Fink's aim is not to divert, nor has he been content to compile a chronological record. He has

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[1] This statement has not the merit of extreme accuracy, for MM. Gauthier-Villars published in 1896 the 2nd edition of a *Cours de Méthodologie Mathématique*, by M. Dauge, pp. 525.

preferred to write for the student, to arouse his attention and stimulate his interest by explaining in general terms the connection of modern researches with the work of the past. And we feel sure that the interests of research will benefit far more from this method of presentation than from a wilderness of biographical matter.

There are natural objections to a treatise dealing separately with the history of the several branches of Mathematical Science, even when that history is confined to Elementary Mathematics. It is exceedingly difficult to draw an arbitrary line between Elementary and Higher Mathematics. As the author says in his preface: "On the one hand certain problems of Elementary Mathematics have from time to time furnished the occasion for the development of higher branches, and on the other from the acquisitions of these new branches, a clear light has fallen on the elementary parts. Accordingly, it may be gratifying . . . . to find here at least that which is fundamental." Of course, by limiting the scope of his treatment to Number-systems and symbols, Arithmetic, Algebra, and Geometry, Dr. Fink has succeeded in reducing what he has to say within reasonable compass. If these limits were much extended, it would require, in these days of specialisation, a Syndicate of Mathematicians to complete a survey at once general and authoritative.

Another objection arises from the difficulty of assigning the boundaries of the branches. We have a series of "spheres of influence," in connection with which we have to face a considerable amount of inter-penetration and a doubtful "hinterland." These obvious objections to his novelty of presentment Dr. Fink meets in his preface, and, on the whole, successfully, and we think that he can fairly claim to have given his readers "a first picture, with fundamental tones clearly sustained, of the principal results" attained in those subjects to which he confines his enquiry.

Dr. Fink, like every other modern historiographer, acknowledges his obligations, and pays his homage to the monumental labours of Cantor, and to the "brilliant, but fragmentary" *Geschichte der Mathematik* of Hankel,[1] "un torse d'une telle beauté qu'il eut été pitié de ne pas le mettre au grand jour."

Turning to detail, it may be worth while to note the following points: p. 137 (l. 8 up) for a read a/b; the statement as to Gauss's theorem on regular polygons on p. 207 is incorrect (*v.* Mathew's *Theory of Numbers*, p. 191), and contradicts a previous reference on p. 161; it was Euler (*Comm. Acad. Scient. Petrop.*, 1738, VI., p. 104) who first corrected Fermat's statement that "all numbers of form 2^m+1, where $m=2^n$ are primes," and not Baltzer, *v.* p. 162; "this point forms the eleventh axiom," p. 270, is a peculiar statement.

On the whole, the translation reads well, but it does not read as easily as did the Klein's *Ausgewählte Fragen der Elementargeometrie.* This, however, may be largely due to the style of the original, and does not diminish our debt to Messrs. Beman and Smith for bringing Dr. Fink's valuable work within the reach of students unacquainted with German.

The 26 pp. of biographical notices—with an average of three lines per item—do not seem to us of much use. What enjoyment can be derived from:

Unger, Ephraim Solomon. Born at Coswig, 1788; died in 1870.
Van Eyck, Jan. 1385-1400. Dutch painter.

In another edition the translators must remember with respect to many names in this list the words of Cantor: "aussi morts que leurs livres; gardons-nous de les ressusciter."

Recueil de Problèmes de Géométrie Analytique, à l'usage des Classes de Mathématiques Spéciales. By F. Michel. Pp. vi. + 240. 1900. Gauthier-Villars.

This collection of exercises in Geometry of Two and Three dimensions contains solutions of the questions set in the Entrance Examinations to the famous *Ecole Polytechnique* from 1860-1900. The solutions are in the main analytical, but in many cases, where analytical methods lead to long and laborious calculations, elegant geometrical solutions are given. The yearly competitions for admission to the great schools—*Ecoles Centrale*, *Polytechnique*, and *Normale*—excite great

[1] What was the date of this? Mr. Rouse Ball gives 1874; Dr. Gow gives it as 1875 (*Hist. Greek Mathcs.*, n. p. 124); Cantor is reported by Miss Scott, last August, as dating it 1876 (*Bull. Am. Math. Soc.*, Nov. 1900.)

interest in French mathematical circles, and solutions appear in most of the journals. M. Michel's collection has one advantage over similar publications of previous years—*e.g.* Rémond (1891), Mosnat (Nony), and Brisse (Gauthier-Villars) 1892—in that a complete bibliography is attached to each solution.

It may interest our readers to see a typical question, set in 1899.

"Indicate in a figure the lines

$$z=2,\ y=-x, \ldots\ldots\ldots\ldots\ldots(\mathrm{A}) \qquad z=-2,\ y=x \ldots\ldots\ldots\ldots\ldots(\mathrm{B})$$

Find the locus of centres S of spheres touching A and B. The locus is a surface (s); find its rectilinear generators. Prove that the coordinate axes are axes of symmetry for (s), and show how this was *a priori* evident.

From each point on (s) deduce a point M, diminishing its ordinate y by $27/(2x+1)^2$. Determine the equation of this new surface (M) and find its generators. Discuss the forms and successive transformations of sections of (M) by planes perpendicular to Oz, and in particular, as completely as possible, sections made by $z=0$, $z=1$."

The solution occupies pp. 219-225 and contains seven figures! *Verb. Sap.*

Traité d'Algèbre. By JOSEPH BERTRAND and H. GARCET. Part I. (17th Edn.). Pp. 324. Part II. (new Edn.). Pp. 388. 1900. (Hachette).

Algebra. By Professor BELLINO CARRARA. Pp. 537. 1900. (Albrighi, Segati). Milano.

Algebra. By WILLIAM THOMSON. Pp. xv. + 560. 1900. (Chambers.)

A book in its 17th edition must appeal strongly to some section of the community, and in the present case the popularity of the two volumes before us will probably be due in no small extent to the fact that they are written up to the syllabus of certain examinations. Not that the book is indifferently written; within its limits it is well enough done. The mere fact that it differs very little from the original edition would not necessarily imply that the *idée mère* of the book is out of harmony with modern views. So far is this from being the case, that I will venture to contrast a passage from the Preface with a passage from that of Chrystal's *Algebra*; the comparison is worth exhibiting as it suggests at least a common source of inspiration.

"La matière (of Algebra) est si abstraite en elle-même, les généralisations, que l'on recontre tout d'abord, sont si importantes pour le succès des études ultérieures; les discussions, à l'aide desquelles on envisage une question sous toutes ses faces, sont si délicates, qu'on ne doit négliger aucun développement, pour initier les élèves aux méthodes et aux procédés de l'*Arithmétique universelle.*"

"The teaching of Algebra in the earlier stages ought to consist in a gradual generalisation of Arithmetic; in other words, Algebra ought, in the first instance, to be taught as *Arithmetica Universalis* in the strictest sense."

The main differences between an elementary text-book in French and in English schools may be briefly stated. Roughly speaking, it consists, perhaps, in the paying of more attention to the "*succès des études ultérieures.*" Hence we find much space devoted to the "solution" of inequalities, to maxima and minima, to questions demanding some considerable knowledge of geometry and mensuration. For instance, the root $x=\infty$ in a simple equation is illustrated from centres of similitude: the 16th question on maxima and minima is "Circumscribe to a sphere a regular truncated pyramid, whose bases are regular octagons. Find the minimum volume of the pyramid as the inclination of the lateral faces to the base varies."

In the second volume we find the difference even more strongly marked. Only 80 pages are given to what we find in our Algebras. The rest is spread over the elements of the Differential Calculus, general Theory of Equations, Elementary Differences and Interpolation, etc. Here again Chap. v. is given up to a collection of those algebraical artifices, a mastery of which is absolutely necessary in analytical geometry of two and three dimensions, *e.g.* the simplification of the general equation in three variables by changing the direction of the coordinate axes; finding the intersections of three surfaces of the second order such that their principal sections are confocal, and so on. Such a book as this is not suited for class work or for the average private student. But the teacher and the clever

boy who is revising work for Scholarship and other purposes will find here a great deal that is both ingenious in treatment and valuable as suggestion and stimulus.

The volumes published by Professor Carrara and Professor Thomson are similar in scope and not very dissimilar in treatment. The Italian Algebra is beautifully printed on well-glazed paper. Professor Thomson's book is the counterpart of Dr. Mackay's *Euclid* and *Arithmetic*, as far as the printer's craft can secure a likeness. Neither has otherwise much to differentiate it, in content, from the ordinary class text-book. Neither makes any use other than incidental of the symbols Σ, Π, which are indispensable in conveying elementary ideas of form and symmetry; indeed Professor Thomson deliberately abstains from their use in the sections dealing with symmetry and cyclical order. Neither uses detached coefficients. Is there any authority for "factoring" and "factored," other than a paper in the *Proc. Edin. Math. Soc.*? The section on graphs is too short to make any serious impression on the mind of the beginner, and maxima and minima receive no systematic treatment, the terms occurring apparently without previous definition (1) in the "variation of a quadratic expression," without graphic illustrations, and (2) in "the maximum value of ${}_nC_r$." The chapter on Indeterminate Coefficients is good. The chief merit of Professor Thomson's book is the excellent arrangement of the sections in the chapters, and the concise and logical statement of the steps in the illustrative solutions. One page is given to Inequalities. At the end of the book this is not enough. Why should not Inequalities of an elementary character, with easy 'Diophantine' questions form a subsection to the first chapter on Equations? Professor Carrara must take it as a compliment if we say that his 'Algebra' is more on the lines of an English text-book than any Continental work we are acquainted with. And, finally, both authors are to be congratulated on their clearness of statement, and their careful and methodical treatment of detail. No good teacher need be afraid to use either book, although neither persistently throws into relief, as it should, the essential requisites in an Algebra—the idea of Algebraic Form.

Exercises in Natural Philosophy. By MAGNUS MACLEAN, D.Sc. Pp. x. + 266. 1900. Longmans, Green, & Co.

This is a carefully graduated series of exercises (with solutions) intended to meet the needs of the candidate for "ordinary degrees," and, as Professor Maclean suggests, to "form a useful supplement to the Elementary Text-books of Physics." As far as the 130 pages devoted to Dynamics are concerned, the candidate for (say) both parts of the London B.Sc. will find much to interest him. The examples touch on Atwood's Machine (with pulleys of given radii and moments of inertia), translational and rotational motion, harmonic motion, governors of steam engines, flywheels, and the gyrostat. Incidentally we notice a solution of a question akin to one proposed in a recent number of the *Ed. Times*: "Find the alteration produced in the length of the day by transporting one hundred million tons of matter from the pole to the equator." The 36 pages of Tables of Physical Constants are the best we have seen, are well up to date, and contain in many cases a most desirable novelty—a column giving the authority. One word more, "conspiring" is a term which gives one pause when applied to "forces."

PROBLEMS.

[*Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.*]

410. [L¹. 4. c.; M¹. 3. b.] On the tangent at M to an ellipse is measured off a length MN equal to the distance of the tangent from the centre of the ellipse: prove that the locus of N is a unicursal whose area is $\frac{\pi}{2}(a+b)^2$.

E. N. BARISIEN.

411. [**A. 3. k.**] Shew that the roots of

$$x^3-\frac{p^2x}{3}-\frac{2p^3}{27}-q^2=0$$

differ by a constant quantity from the squares of the corresponding roots of

$$x^3+px+q=0.$$

E. BUDDEN.

412. [**L^1. 2; 14. a.**] E is a conic circumscribing the triangle ABC, and its centre O lies on a fixed straight line: shew that the poles of the sides of ABC with respect to E lie on three conics which cut the corresponding sides in points on a fourth conic. E. P. EVANS.

413. [**K.**] Along three straight lines meeting in O at angles α, β, γ, lengths x, y, z, are measured: prove that the three points so obtained lie on a straight line if

$$\Sigma x^{-1}\sin\alpha=0,$$

and on a circle through O if $\quad \Sigma x\sin\alpha=0.$ R. W. H. T. HUDSON.

414. [**K. 2. b. c.**] Find the angle made with BC by the common tangent of the nine-point circle and the ex-circle opposite A, and the ratio in which it divides the sides of the triangle ABC. H. G. M

415. [**K. 9. b.**] It is required to inscribe in a given circle $x^2+y^2=a^2$ a regular heptagon, one of whose angular points lies at (a, o). Prove that the two rectangular hyperbolas whose centres are at the points $(\frac{1}{4}a, \pm\frac{1}{4}a\sqrt{7})$, whose transverse axes are parallel to Ox, and which pass through the point (a, o), cut the circle also in the required angular points. [Suggestions for simpler constructions are invited.] H. RICHMOND.

416. [**M^1. 2. a.**] Find on a given straight line two points which shall be homologues in two given similar figures. C. E. YOUNGMAN.

417. [**L^1. 5; 17. e.**] Eight normals to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touch any concentric co-axial ellipse $\frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1$. Prove this, and find the conditions for reality. Trip., 1895.

418. [**R. 7. f.**] A circular wire of radius a moves in its own plane without rotation, so that its centre has a simple harmonic motion of amplitude a; a bead moves on the wire uniformly, completing a circuit in the period of the simple harmonic motion, and being in the line of the motion of the centre when the centre is in its mean position: prove that the acceleration of the bead is towards the centre of the simple harmonic motion, and that its path is an ellipse of eccentricity $(\frac{2}{5}\sqrt{5}-\frac{4}{5})^{\frac{1}{2}}$. Pemb. (C.), 1897.

419. [**R. 9. b.**] A, B are two smooth holes in a smooth horizontal table, distance $2a$ apart; a particle of mass M rests on the table midway between A and B, and a particle of mass m hangs beneath the table suspended from M by two equal weightless inextensible strings, each of length $a(1+\sec\alpha)$, passing through A and B; a blow J is given to M in a direction perpendicular to AB. Shew that if $J^2>2Magm\tan\alpha$, M will oscillate to and fro through a distance $2a\tan\alpha$, but if J^2 is less than this quantity and $=2Mmag(\tan\alpha-\tan\beta)$, the distance through which M oscillates will be $2a\sqrt{(\sec\alpha-\sec\beta)(\sec\alpha-\sec\beta+2)}$. St. John's (C.), 1895.

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 285, 326, 336-8, 341, 349, 356, 369, 370, 373-82, 387, 389, 393, 395-7, 401, 404, 406, 408-9.

Solutions to the above, or other questions to which no solution has yet been published, and to 410-19 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be **very carefully drawn to a small scale** on a separate sheet.

51. [K. 15. a. I. 1.] *If the diameter of a cylinder be d inches and its height h yards, and the s.g. of its material s, show that approximately weight in lbs.* $=sd^2h(1+\frac{1}{50})$; *volume in gals.* $=\frac{1}{10}d^2h(1+\frac{1}{50})$. *Hence find weight of solid sphere one foot in diameter, s.g.* $=7{\cdot}8$. A. LODGE.

Solution.

Volume of cylinder $=d^2h\,.\,\frac{\pi}{4}\,.\,(\frac{1}{12})^2\,.\,3$ cub. ft.

No. of gallons $=d^2h\frac{3\pi}{4{\cdot}144}\times 6(1+\frac{1}{30}+\frac{1}{200})$ gals.

$=d^2h\,.\,\frac{1}{10}(1{\cdot}0194)$ gals.

$=\frac{1}{10}d^2h(1+\frac{1}{50})$ gals. approx.

weight in lbs. $=sd^2h(1+\frac{1}{50})$ lbs.

If the diameter of the sphere be 1 ft., that of the equivalent cylinder is $\sqrt{\frac{2}{3}}$ ft.;

$\therefore$ weight of sphere is $\frac{1}{3}\times\frac{2}{3}\times 144(1+\frac{1}{50})$ lbs. $\times 7{\cdot}8$ lbs.

$=32{\cdot}64\times 7{\cdot}8$ lbs. $=254$ lbs. approx.

53. K. 9. b.] *A square and a regular hexagon of equal area are taken; in the square is inscribed a circle, in this circle a square, and so on. In the hexagon is inscribed a circle, in this circle a hexagon, and so on. Prove that the sum of the areas of the hexagons is double that of the squares.* E. M. LANGLEY.

Solution by E. Ll. TANNER.

If the side of the square be a, and that of the hexagon be x, then

$$a^2=\frac{3\sqrt{3}}{2}x^2;$$

the sides of the squares are $a,\ \frac{a}{\sqrt{2}},\ \frac{a}{(\sqrt{2})^2},\ \dots;$

the sum of the areas $=a^2\left(1+\frac{1}{2}+\frac{1}{2^2}+\dots\right)=2a^2;$

the sides of the hexagons are $x,\ \frac{x\sqrt{3}}{2},\ x\left(\frac{\sqrt{3}}{2}\right)^2,\ \dots;$

the sum of the areas $=\frac{3x^2\sqrt{3}}{2}\left(1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\dots\right)=6x^2\sqrt{3}.$

But $6x^2\sqrt{3}=4a^2=2$(sum of areas of squares); $\therefore$ etc.

54. (Corrected.) [K. 11. b.] *From the mid point O of the line joining the centres of two given intersecting circles are drawn OK perpendicular to a common tangent, and OL perpendicular to any other straight line passing through the external centre of similitude. If OL meets the common chord in F, show that*

$OF : OL = OK^2 : OL^2$. Hence show the locus of the mid points of parallel chords of a conic is a straight line. E. M. LANGLEY.

Solution by PROPOSER and W. J. GREENSTREET.

If S be one of the intersections of the circles, the common chord bisects the common tangent in K, and meets any other straight line RL through R the external c.s. in D. Then OL perpendicular to RD cuts SD in F.

The triangles OKL, OKF are similar,

$$\therefore\ OF/OK = OK/OL\ ; \quad \therefore\ OF/OL = OK^2/OL^2.$$

If P, Q the centres of the circles be the ends of one of a set of parallel chords in a conic, focus S, directrix RD,

$OK = \frac{1}{2}(SP+SQ) = e\,.\,OL$; $\therefore\ OF = e^2 OL$; but $CS = e^2\,.\,CX$, and D is fixed ; $\therefore\ O$ lies on a fixed line CD.

56. (Corrected.) [K. 10. e.] *From a point S on a circle SPM a perpendicular is drawn to the diameter through P meeting any other chord PM through P in K. Show that the triangles SPK, SPM are similar. Hence show that in a central conic SY, CP intersect on the directrix.* E. M. LANGLEY.

Solution.

Let the perpendicular meet the diameter PZ in Y, and let the chord MZ meet the tangent at S in X. Let XM, SK meet in U, and UP meet XS in C.

$$S\hat{M}P = S\hat{Z}P = 90^\circ - S\hat{P}Z = P\hat{S}K\ ;$$

$S\hat{P}M$ is common to the triangles SPM, SPK.

$$\therefore\ PK/SP = SP/PM.$$

If the letters be taken as if in a diagram in conics [so that PM is $\parallel$ to SX],

$$PK = e^2 PM\ ; \text{ but } CS = e^2 CX.$$

$\therefore\ SK$ (the perpendicular on the tangent at P) and CP (the diameter through P) intersect on XM (the directrix).

58. [K. 2. a.] *Construct the points whose pedal lines with respect to a given triangle pass through a given point, and show that there are in general three solutions.* W. C. FLETCHER.

Solution by R TUCKER.

Let P be a point on the circumcircle of ABC ; $P\hat{B}A = \theta$; and let DEF be the Wallace line of P.

The equation to DEF is

$$a\alpha \cos \overline{C-\theta} \cos \overline{B+\theta} \sin \theta - b\beta \cos\theta \sin \overline{C-\theta} \cos \overline{B+\theta}$$
$$+ c\gamma \cos \overline{C-\theta} \sin \overline{B+\theta} \cos \theta = 0, \quad \text{.........(1)}$$

or

$$a\alpha(\cos C + \sin C \tan\theta)(\cos B - \sin B \tan\theta)\tan\theta$$
$$- b\beta(\sin C - \cos C \tan\theta)(\cos B - \sin B \tan\theta)$$
$$+ c\gamma(\cos C + \sin C \tan\theta)(\sin B + \cos B \tan\theta) = 0.$$

Similar equations give the Wallace lines of Q and R on the circumcircle.

For the point required we have a determinant of the form

$$\begin{vmatrix} A & B & C \\ A' & B' & C' \\ \begin{matrix} t(\cos C + t\sin C) \\ \times(\cos B - t\sin B) \end{matrix} & \begin{matrix} -(\sin C - t\cos C) \\ \times(\cos B - t\sin B) \end{matrix} & \begin{matrix} (\cos C + t\sin C) \\ \times(\sin B + t\cos B) \end{matrix} \end{vmatrix} = 0$$

in general giving a cubic for $\tan\theta$.

61. [K. 8. d.] *In the trapezoid ABCD, the parallel sides AB, CD are constants, and meet the variable base BD at right angles; prove that the perpendicular to BD from the intersection of the diagonals AD, BC is constant.*

PROF. D. S. WRIGHT.

Solution.

Let OE be the parallel to AB (a) or CD (b) through O the intersection of diagonals, cutting BD in E.

Then $$\frac{OE}{b}=\frac{BE}{BD};\quad \frac{OE}{a}=\frac{DE}{BD}.$$

$$\therefore\ BD\,.\,OE\left(\frac{1}{a}+\frac{1}{b}\right)=BD,\quad\text{or}\quad OE=\frac{ab}{a+b}=\text{const.}$$

62. [A. 1. a.] *Simplify*

$$\frac{1}{1-x^4}\left\{1+\frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{(1-x)(1-x^2)}+\frac{x^3}{(1-x^2)(1-x^3)}\right\}.$$

Solution.

$$[(1-x)(1-x^2)(1-x^3)]^{-1}.$$

63. [K. 8. e.] *Show that a rectangle of given perimeter has the greatest area when its sides are equal.*

Solution.

If a, b are the sides, we require the maximum value of ab, where

$$a+b=\text{constant}.$$

Now $$ab=\tfrac{1}{4}[(a+b)^2-(a-b)^2]=\text{max. when } a=b.$$

64. [K. 10. e.] *From a given point O without a circle draw, when possible, a straight line which shall have its middle point and its other end on the circumference.*

Solution.

With centre O and radius $OT\sqrt{2}$, where OT is the tangent from the given point O to the circle, describe a circle cutting the given circle at B.

Let OB cut the first circle in A. $OA\,.\,OB=OT^2$. $\therefore\ 2OA=OB$.

The problem is not possible if the circles do not meet.

65. [K. 10. e.] *The sum of the squares of the distances of a fixed point on a circle from the ends of a variable chord is constant; show that the locus of the middle point of the chord is a straight line.*

Solution.

Let AB be the variable chord, M its mid point; C the fixed point, O the centre of the circle, MP perpendicular to CO at P.

$$\text{Given constant}=CA^2+CB^2=2CM^2+2MB^2=2CM^2+2OB^2-2OM^2.$$

$$\therefore\ CM^2-OM^2=\text{const.}=CP^2-OP^2.\quad \therefore\ P\text{ is fixed.}$$

$\therefore$ locus of M is a perpendicular to CO, through P.

66. [K. 20. d.] *Prove* $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{63}{65}$.

Solution.

If $\sin A=\frac{4}{5}$, $\sin B=\frac{5}{13}$, then $\cos A=\frac{3}{5}$, $\cos B=\frac{12}{13}$.

$$\therefore\ \text{L.H.}=\sin^{-1}(\tfrac{4}{5}\cdot\tfrac{12}{13}+\tfrac{5}{13}\cdot\tfrac{3}{5})=\sin^{-1}\tfrac{63}{65}.$$

68. [R. 1. d.] *From a ship sailing N.E. at 15 miles per hour a second ship is observed to be always due S. If the second ship is sailing E., find the rate at which it is travelling.*

Solution.

If v be required rate, then, since the line joining the ships is always north and south, $v=15\cos 45°$.

$$\therefore\ v=15/\sqrt{2} \text{ miles per hour.}$$

72. [K. 10. e.] *Through the vertex A of a square ABCD a straight line is drawn meeting CB, CD, DB at E, F, K respectively; prove that CK touches the circle CEF.*

Solution.

K is a point on the diagonal, $\therefore$ the triangles DAK, DCK are congruent.

$$\therefore\ K\hat{C}F=K\hat{A}D=C\hat{E}F. \quad \therefore\ CK \text{ is a tangent to the circle } ECF.$$

75. [L[1]. 11. b.] *PQ, PR are diameter and normal to a rectangular hyperbola. Prove that PQR is a right angle.*

v. Besant (1895), Prop. VII., p. 129.

78. [L[1]. 17. a.] *A chord PQ of a conic passes through a fixed point. If the circle on PQ as diameter meet the conic again in P′, Q′, show that P′Q′ also passes through a fixed point.*

Solution.

Let $a\xi(x-\xi)+b\eta(y-\eta)=0$; $(x-\xi)^2+(y-\eta)^2=r^2$, be chord and circle.

Then
$$ax^2+by^2-1+\lambda(\overline{x-\xi}^2+\overline{y-\eta}^2-r^2)=0$$
is the same pair of lines as
$$(a\xi x+by\eta-a\xi^2-b\eta^2)(a\xi x-by\eta+k)=0$$
giving
$$k=(a\xi^2+b\eta^2)(a+b)/(a-b).$$

If f, g be the fixed point,
$$a\xi f+b\eta g=a\xi^2+b\eta^2,$$
$$\therefore\ a\xi x-b\eta y+\frac{a+b}{a-b}(a\xi^2+b\eta^2)=0$$
passes through the fixed point
$$x/f=y/-g=-\frac{a+b}{a-b}.$$

98. [J. 2. f.] *If n^2 coins, of which exactly n are silver, are arranged at random in n rows, each containing n coins, the chance that one row at least occurs in which there is no silver coin is* $1-\dfrac{(n-1)!(n^2-n)!\,n^{n-1}}{(n^2-1)!}$.

King's and Pem., 1895

Solution by PROFESSOR A. LODGE.

If n coins are chosen at random out of n^2 coins, of which n are silver and n^2-n are not, the chance the first coin is silver and the others not is
$$\frac{n}{n^2}\cdot\frac{n^2-n}{n^2-1}\cdot\frac{n^2-n-1}{n^2-2}\cdots\frac{n^2-2n+2}{n^2-n+1}=u_n.$$

The chance that the second is silver and the rest not is
$$\frac{n^2-n}{n^2}\cdot\frac{n}{n^2-1}\cdot\frac{n^2-n-1}{n^2-2}\cdots\frac{n^2-2n+2}{n^2-n+1}, \text{ which also } =u_n.$$

$\therefore\ nu_n$ is the chance that just one is silver and the rest not.

If this condition be satisfied, the chance that a similar condition shall be satisfied in a second set of n coins is

$$n\,.\,\frac{n-1}{n^2-n}\,.\,\frac{n^2-2n+1}{n^2-n-1}\,.\,\frac{n^2-2n}{n^2-n-2}\cdots\frac{n^2-3n+3}{n^2-2n+1}=nv_n,$$

and n^2-2n will be left, $n-2$ silver and n^2-3n+2 not silver.

$\therefore$ the chance that this condition shall be satisfied in the first $n-1$ rows is $n^{n-1}u_nv_n\ldots$, which reduces to

$$n^{n-1}(n-1)!(n^2-n)!/(n^2-1)!$$

The chance we require, of failure in any row, is

$$1-n^{n-1}(n-1)!(n^2-n)!/(n^2-1)!$$

111. [R. 4. c.] *A triangle ABC, right-angled at C, formed of three uniform rods jointed at their ends is suspended by a string attached to the middle point of AB. Show that the reaction at C is $cw/(2\sqrt{2})$, where w is the weight per unit length of each rod.* Queen's (C.), 1896.

Solution by Professor A. Lodge.

Let DG, EH, FK be the verticals through the middle points of the rods a, b, c respectively, and let AHK, BGK, and GCH be the lines of the reactions at the joints, which must meet two by two on the above verticals, to secure the equilibrium of the separate rods.

Let the vertical lines AM, BL be drawn to meet GCH in L and M.

The rods AC, BC must make equal angles (45°) with the vertical, since then there will be equal balancing portions of the three rods on either side of the line of support FK. Hence $LC : CM = a : b$.

The triangles AHM, BLG are force-triangles for the rods AC, BC respectively. Hence, denoting the reaction at C by r,

$$\frac{BL}{LG}=\frac{aw}{r}\,;\quad\text{and}\quad\frac{AM}{HM}=\frac{bw}{r}\,;\quad\text{and}\quad\frac{LG}{HM}=\frac{LC}{CM}=\frac{a}{b}.$$

$$\therefore\quad\frac{BL}{AM}=\frac{a^2}{b^2}.$$

This enables the line of action of r to be drawn.

For, if CP be drawn perpendicular to AB, $BP : PA = a^2 : b^2$, and therefore, if GCH be drawn making an angle α with CB equal to the angle PCB, cutting AB produced in Q, we shall have $QB : QA = a^2 : b^2$, and therefore $BL : AM = a^2 : b^2$, which is the required ratio.

Now we can determine r; for

$$\frac{aw}{r}=\frac{BL}{LG}=\frac{2BL}{LC}=\frac{2\sin\alpha}{\sin 45^\circ}=2\sqrt{2}\sin BAC=\frac{2\sqrt{2}\,.\,a}{c}.$$

$$\therefore\quad r=\frac{cw}{2\sqrt{2}}.$$

Note.—It can be easily proved analytically, by taking CA, CB as axes of coordinates, that PC produced passes through K, and that $CK=CF$. Either of these theorems gives an easy way of finding K, *i.e.* of finding the lines of the reactions at A and B.

126. [K. 8. e.] *Find the minimum rhombus that can be cut out of a given parallelogram.*

Solution by Professor A. Lodge.

Let O be the centre of the parallelogram, and let OA, OB be drawn parallel to the sides, the angle between them being ω.

The diagonals of any inscribed rhombus must pass through O and be at

right angles to each other. Let OP, OQ be any position of these diagonals, and let the angle $POA=\theta$. The area of the rhombus is $2OP \,.\, OQ$, hence $OP \,.\, OQ$ has to be a minimum.

The angle $OPA=\pi-(\omega+\theta)$, and the angle $OQB=\frac{\pi}{2}+\theta$.

$$\therefore \frac{OP}{OA}=\frac{\sin\omega}{\sin(\omega+\theta)}; \text{ and } \frac{OQ}{OB}=\frac{\sin\omega}{\cos\theta};$$

$$\therefore \frac{OP \,.\, OQ}{OA \,.\, OB}=\frac{\sin^2\omega}{\sin(\omega+\theta)\cos\theta}=\frac{2\sin^2\omega}{\sin(\omega+2\theta)+\sin\omega},$$

which is a minimum when $\omega+2\theta=\frac{\pi}{2}$, *i.e.* when $2\theta=\frac{\pi}{2}-\omega$.

Hence, if OM, ON are drawn perpendicular to AP, BQ respectively, the diagonals of the minimum rhombus will bisect the angles AOM, BON respectively.

The ratio of the area of the minimum rhombus to that of the given parallelogram

$$=\frac{2OP \,.\, OQ}{4OA \,.\, OB\sin\omega}=\frac{\sin\omega}{1+\sin\omega},$$

and is greatest when $\omega=90°$.

Note.—The solution assumes that AP will be less than OB, and BQ less than OA. If this be not the case the minimum rhombus will have for one of its diagonals the shorter diagonal of the parallelogram.

142. [M^4. e.] *Two variable lines through fixed points make angles θ, ϕ with any two fixed directions respectively; trace the curve described by their point of intersection when*

$$e^{a\theta}\sin\theta=e^{a\phi}\sin\phi.$$

Solution by Professor A. Lodge.

One solution is obviously $\theta=\phi$. Hence, if the fixed points are A, B, and if AC, BC are drawn in the fixed directions to meet in C, one locus will be the circum-circle of ABC. As many solutions as desired may be obtained by drawing $r=e^{a\theta}\sin\theta$; (1) with origin A, initial line AC; (2) with origin B and initial line BC.

Take any point P on the first curve, and with centre B and radius $BQ=AP$ draw a circle cutting the second curve in Q_1, Q_2, ... Then the intersections of AP with BQ_1, BQ_2, ... are all points on different loci satisfying

$$e^{a\theta}\sin\theta=e^{a\phi}\sin\phi.$$

Taking different points P, we get a series of points Q_1, Q_2, through which the loci of Q_1, Q_2, ... can be drawn.

[It would probably be simpler to draw $r=b\theta+\log_{10}\sin\theta$ as foundation curves, where $b=\frac{\pi a}{180}\log_{10}e$ if θ is measured in degrees.]

366. [K. 3. b.] *ABC, DBC are two equilateral triangles on the same base BC. A point P is taken on the circle DBC. Show that PA, PB, PC are the sides of a right-angled triangle.* E. M. Radford.

Solution.

$$PB^2+PC^2=2PO^2+2OB^2, \text{ add to each side } 2OD^2+2OB^2;$$

$$\therefore PB^2+PC^2+2OD^2+2OB^2=2PO^2+2OB^2+2OD^2+2OB^2.$$

$$\therefore PB^2+PC^2+2BD^2=PA^2+PD^2+BC^2. \quad \because 2BO=BC.$$

$$\therefore PB^2+PC^2=PA^2.$$

380. [L[1]. 11. c.] (1) *A chord of a rectangular hyperbola subtends a right angle at a fixed point. Find geometrically the envelope of the chord.*

E. N. Barisien.

Solution by W. F. Beard.

The locus of the point of intersection of orthogonal tangents to a conic is a circle.

Reciprocate with regard to any point on this circle. The reciprocal of the conic is a rectangular hyperbola; the reciprocal of the circle is a parabola; and the theorem becomes therefore:—chords of a rectangular hyperbola which subtend a right angle at a fixed point, touch a fixed parabola with the fixed point as focus.

383. [D. 2. d.] *Prove that*

$$\left(\underset{\cdot}{a}+\frac{1}{b+}\,\frac{1}{\underset{\cdot}{c}+}\,\frac{1}{a+}\dots\right)\left(\underset{\cdot}{b}+\frac{1}{c+}\,\frac{1}{\underset{\cdot}{a}+}\,\frac{1}{b+}\dots\right)\left(\underset{\cdot}{c}+\frac{1}{a+}\,\frac{1}{\underset{\cdot}{b}+}\,\frac{1}{c+}\dots\right)=\left(\underset{\cdot}{t}+\frac{1}{t+}\dots\right),$$

where $t=\Sigma a+abc.$

R. W. Genese.

Solution by W. F. Beard.

Let $x=\underset{\cdot}{a}+\frac{1}{b+}\,\frac{1}{\underset{\cdot}{c}+}\,\frac{1}{a+}\dots,\quad y=\underset{\cdot}{b}+\frac{1}{c+}\,\frac{1}{\underset{\cdot}{a}+}\,\frac{1}{b+}\dots,\quad z=\underset{\cdot}{c}+\frac{1}{a+}\,\frac{1}{\underset{\cdot}{b}+}\,\frac{1}{c+}\dots,$

and $t=a+b+c+abc.$

$$x=a+\frac{1}{y} \text{ or } x-a=\frac{1}{y}.$$

Similarly

$$\left.\begin{aligned} y-b&=\frac{1}{z}\\ z-c&=\frac{1}{x}.\end{aligned}\right\}\quad\dots\dots\dots\dots\text{(i.)}$$

$$\therefore\ (x-a)(y-b)(z-c)=\frac{1}{xyz};$$

or $$xyz-ayz-bzx-cxy+bcx+cay+abz-abc=\frac{1}{xyz};$$

but $$cay-cxy=cy(a-x)=-c \text{ from (i.)},$$

similarly $$bcx-bzx=-b,$$

and $$abz-ayz=\qquad -a;$$

$\therefore$ we have $$xyz-a-b-c-abc=\frac{1}{xyz},$$

$$xyz=t+\frac{1}{xyz}$$

$$=t+\frac{1}{t+}\,\frac{1}{t+}\dots.$$

Q.E.D.

386. [L[1]. 3. c.] *If a circle through the centre of an ellipse cut pairs of conjugate diameters in A, A'; B, B'; ... then shall the chords AA', BB', ... all pass through a fixed point.*

A. Lodge.

Solution by W. F. Beard.

Conjugate diameters form a pencil in involution. Thus the pencil $C(AA'BB'\dots)$ is in involution, and therefore (*vide* Russell's *Pure Geom.*, ch. xx., § 2), AA', BB', etc., are concurrent.

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* Will be reviewed shortly.

ERRATA.

Jan. 1901, front page cover, *for* Vol. II., No. 1 *read* Vol. II., No. 25.

p. 3, l. 20, up, *for* $-qr$ *read* $-9r$.

p. 3, l. 9, up, *for* $+qr^2$ *read* $+9r^2$.

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